

# ON INCOMPACTNESS FOR CHROMATIC NUMBER OF GRAPHS

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**ABSTRACT.** We deal with compactness. Assume the existence of non-reflecting stationary set of cofinality  $\kappa$ . We prove that one can define a graph  $G$  whose chromatic number is  $> \kappa$ , while the chromatic number of every subgraph  $G' \subseteq G, |G'| < |G|$  is  $\leq \kappa$ . The main case is  $\kappa = \aleph_0$ .

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*Date:* June 28, 2012.

*2010 Mathematics Subject Classification.* Primary: 03E05; Secondary: 05C15.

*Key words and phrases.* set theory, graphs, chromatic number, compactness, non-reflecting stationary sets.

The author thanks Alice Leonhardt for the beautiful typing.

The author would like to thank the Israel Science Foundation for partial support of this research (Grant no. 1053/11). Publication 1006.

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[We show that “ $S \subseteq S_\kappa^\lambda$  is stationary not reflecting” implies compactness for length  $\lambda$  for “chromatic number =  $\kappa$ ”.]

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[Here we weaken the assumption in §1 to “ $\mathcal{A} \subseteq {}^\kappa \text{Ord}$  is almost free”.]

## § 0. INTRODUCTION

§ 0(A). **The questions and results.** During the Hajnal conference (June 2011) Magidor asked me on incompactness of “having chromatic number  $\aleph_0$ ”; that is, there is a graph  $G$  with  $\lambda$  nodes, chromatic number  $> \aleph_0$  but every subgraph with  $< \lambda$  nodes has chromatic number  $\aleph_0$  when:

- (\*)<sub>1</sub>  $\lambda$  is regular  $> \aleph_1$  with a non-reflecting stationary  $S \subseteq S_{\aleph_0}^\lambda$ , possibly though better not, assuming some version of GCH.

Subsequently also when:

- (\*)<sub>2</sub>  $\lambda = \aleph_{\omega+1}$ .

Such problems were first asked by Erdős-Hajnal, see [EH74]; we continue [Sh:347].

First answer was using BB, see [Sh:309, 3.24] so assuming

- ⊞ (a)  $\lambda = \mu^+$
- (b)  $\mu^{\aleph_0} = \mu$
- (c)  $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \aleph_0\}$  is stationary not reflecting

or just

- ⊞' (a)  $\lambda = \text{cf}(\lambda)$
- (b)  $\alpha < \lambda \Rightarrow |\alpha|^{\aleph_0} < \lambda$
- (c) as above.

However, eventually we get more: if  $\lambda = \lambda^{\aleph_0} = \text{cf}(\lambda)$  and  $S \subseteq S_{\aleph_0}^\lambda$  is stationary non-reflective then we have  $\lambda$ -incompactness for  $\aleph_0$ -chromatic. In fact, we replace  $\aleph_0$  by  $\kappa = \text{cf}(\kappa) < \lambda$  using a suitable hypothesis.

Moreover, if  $\lambda^\kappa > \lambda$  we still get  $(\lambda^\kappa, \lambda)$ -incompactness for  $\kappa$ -chromatic number. In §2 we use quite free family of countable sequences.

In subsequent work we shall solve also the parallel of the second question of Magidor, i.e.

- (\*)<sub>2</sub> for regular  $\kappa \geq \aleph_0$  and  $\varepsilon < \kappa$  there is a graph  $G$  of chromatic number  $> \kappa$  but every sub-graph with  $< \aleph_{\kappa \cdot \varepsilon + 1}$  nodes has chromatic number  $\leq \kappa$ .

We thank Menachem Magidor for asking, Peter Komjath for stimulating discussion and Paul Larson, Shimoni Garti and the referee for some comments.

## § 0(B). Preliminaries.

**Definition 0.1.** For a graph  $G$ , let  $\text{ch}(G)$ , the chromatic number of  $G$  be the minimal cardinal  $\chi$  such that there is colouring  $\mathbf{c}$  of  $G$  with  $\chi$  colours, that is  $\mathbf{c}$  is a function from the set of nodes of  $G$  into  $\chi$  or just a set of of cardinality  $\leq \chi$  such that  $\mathbf{c}(x) = \mathbf{c}(y) \Rightarrow \{x, y\} \notin \text{edge}(G)$ .

**Definition 0.2.** 1) We say “we have  $\lambda$ -incompactness for the  $(< \chi)$ -chromatic number” or  $\text{INC}_{\text{chr}}(\lambda, < \chi)$  when: there is a graph  $G$  with  $\lambda$  nodes, chromatic number  $\geq \chi$  but every subgraph with  $< \lambda$  nodes has chromatic number  $< \chi$ .

2) If  $\chi = \mu^+$  we may replace “ $< \chi$ ” by  $\mu$ ; similarly in 0.3.

We also consider

- Definition 0.3.** 1) We say “we have  $(\mu, \lambda)$ -incompactness for  $(< \chi)$ -chromatic number” or  $\text{INC}_{\text{chr}}(\mu, \lambda, < \chi)$  when there is an increasing continuous sequence  $\langle G_i : i \leq \lambda \rangle$  of graphs each with  $\leq \mu$  nodes,  $G_i$  an induced subgraph of  $G_\lambda$  with  $\text{ch}(G_\lambda) \geq \chi$  but  $i < \lambda \Rightarrow \text{ch}(G_i) < \chi$ .
- 2) Replacing (in part (1))  $\chi$  by  $\bar{\chi} = (< \chi_0, \chi_1)$  means  $\text{ch}(G_\lambda) \geq \chi_1$  and  $i < \lambda \Rightarrow \text{ch}(G_i) < \chi_0$ ; similarly in 0.2 and parts 3), 4) below.
- 3) We say we have incompactness for length  $\lambda$  for  $(< \chi)$ -chromatic (or  $\bar{\chi}$ -chromatic) number when we fail to have  $(\mu, \lambda)$ -compactness for  $(< \chi)$ -chromatic (or  $\bar{\chi}$ -chromatic) number for some  $\mu$ .
- 4) We say we have  $[\mu, \lambda]$ -incompactness for  $(< \chi)$ -chromatic number or  $\text{INC}_{\text{chr}}[\mu, \lambda, < \chi]$  when there is a graph  $G$  with  $\mu$  nodes,  $\text{ch}(G) \geq \chi$  but  $G^1 \subseteq G \wedge |G^1| < \lambda \Rightarrow \text{ch}(G^1) < \chi$ .
- 5) Let  $\text{INC}_{\text{chr}}^+(\mu, \lambda, < \chi)$  be as in part (1) but we add that even the  $\text{cl}(G_i)$ , the colouring number of  $G_i$  is  $< \chi$  for  $i < \lambda$ , see below.
- 6) Let  $\text{INC}_{\text{chr}}^+[\mu, \lambda, < \chi]$  be as in part (4) but we add  $G^1 \subseteq G \wedge |G^1| < \lambda \Rightarrow \text{cl}(G^1) < \chi$ .
- 7) If  $\chi = \kappa^+$  we may write  $\kappa$  instead of “ $< \chi$ ”.

- Definition 0.4.** 1) For regular  $\lambda > \kappa$  let  $S_\kappa^\lambda = \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ .
- 2) We say  $C$  is a  $(\geq \theta)$ -closed subset of a set  $B$  of ordinals when: if  $\delta = \sup(\delta \cap B) \in B$ ,  $\text{cf}(\delta) \geq \theta$  and  $\delta = \sup(C \cap \delta)$  then  $\delta \in C$ .

**Definition 0.5.** For a graph  $G$ , the colouring number  $\text{cl}(G)$  is the minimal  $\kappa$  such that there is a list  $\langle a_\alpha : \alpha < \alpha(*) \rangle$  of the nodes of  $G$  such that  $\alpha < \alpha(*) \Rightarrow \kappa > |\{\beta < \alpha : \{a_\beta, a_\alpha\} \in \text{edge}(G)\}|$ .

§ 1. FROM NON-REFLECTING STATIONARY IN COFINALITY  $\aleph_0$ 

**Claim 1.1.** *There is a graph  $G$  with  $\lambda$  nodes and chromatic number  $> \kappa$  but every subgraph with  $< \lambda$  nodes have chromatic number  $\leq \kappa$  when:*

- ⊞ (a)  $\lambda, \kappa$  are regular cardinals
- (b)  $\kappa < \lambda = \lambda^\kappa$
- (c)  $S \subseteq S_\kappa^\lambda$  is stationary, not reflecting.

*Proof. Stage A:* Let  $\bar{X} = \langle X_i : i < \lambda \rangle$  be a partition of  $\lambda$  to sets such that  $|X_i| = \lambda$  or just  $|X_i| = |i + 2|^\kappa$  and  $\min(X_i) \geq i$  and let  $X_{<i} = \cup\{X_j : j < i\}$  and  $X_{\leq i} = X_{<(i+1)}$ . For  $\alpha < \lambda$  let  $\mathbf{i}(\alpha)$  be the unique ordinal  $i < \lambda$  such that  $\alpha \in X_i$ . We choose the set of points = nodes of  $G$  as  $Y = \{(\alpha, \beta) : \alpha < \beta < \lambda, \mathbf{i}(\beta) \in S \text{ and } \alpha < \mathbf{i}(\beta)\}$  and let  $Y_{<i} = \{(\alpha, \beta) \in Y : \mathbf{i}(\beta) < i\}$ .

*Stage B:* Note that if  $\lambda = \kappa^+$ , the complete graph with  $\lambda$  nodes is an example (no use of the further information in ⊞). So without loss of generality  $\lambda > \kappa^+$ .

Now choose a sequence satisfying the following properties, exists by [Sh:g, Ch.III]:

- ⊞ (a)  $\bar{C} = \langle C_\delta : \delta \in S \rangle$
- (b)  $C_\delta \subseteq \delta = \sup(C_\delta)$
- (c)  $\text{otp}(C_\delta) = \kappa$  such that  $(\forall \beta \in C_\delta)(\beta + 1, \beta + 2 \notin C_\delta)$
- (d)  $\bar{C}$  guesses<sup>1</sup>clubs.

Let  $\langle \alpha_{\delta, \varepsilon}^* : \varepsilon < \kappa \rangle$  list  $C_\delta$  in increasing order.

For  $\delta \in S$  let  $\Gamma_\delta$  be the set of sequence  $\bar{\beta}$  such that:

- ⊞ $\bar{\beta}$  (a)  $\bar{\beta}$  has the form  $\langle \beta_\varepsilon : \varepsilon < \kappa \rangle$
- (b)  $\bar{\beta}$  is increasing with limit  $\delta$
- (c)  $\alpha_{\delta, \varepsilon}^* < \beta_{2\varepsilon+i} < \alpha_{\delta, \varepsilon+1}^*$  for  $i < 2, \varepsilon < \kappa$
- (d)  $\beta_{2\varepsilon+i} \in X_{<\alpha_{\delta, \varepsilon+1}^*} \setminus X_{\leq \alpha_{\delta, \varepsilon}^*}$  for  $i < 2, \varepsilon < \kappa$
- (e)  $(\beta_{2\varepsilon}, \beta_{2\varepsilon+1}) \in Y$  hence  $\in Y_{<\alpha_{\delta, \varepsilon+1}^*} \subseteq Y_{<\delta}$  for each  $\varepsilon < \kappa$

(can ask less).

So  $|\Gamma_\delta| \leq |\delta|^\kappa \leq |X_\delta| \leq \lambda$  hence we can choose a sequence  $\langle \bar{\beta}_\gamma : \gamma \in X'_\delta \subseteq X_\delta \rangle$  listing  $\Gamma_\delta$ .

Now we define the set of edges of  $G$ :  $\text{edge}(G) = \{(\alpha_1, \alpha_2), (\min(C_\delta), \gamma)\} : \delta \in S, \gamma \in X'_\delta \text{ hence the sequence } \bar{\beta}_\gamma = \langle \beta_{\gamma, \varepsilon} : \varepsilon < \kappa \rangle \text{ is well defined and we demand } (\alpha_1, \alpha_2) \in \{(\beta_{\gamma, 2\varepsilon}, \beta_{\gamma, 2\varepsilon+1}) : \varepsilon < \kappa\}\}$ .

*Stage C:* Every subgraph of  $G$  of cardinality  $< \lambda$  has chromatic number  $\leq \kappa$ .

For this we shall prove that:

$$\oplus_1 \text{ ch}(G|Y_{<i}) \leq \kappa \text{ for every } i < \lambda.$$

This suffice as  $\lambda$  is regular, hence every subgraph with  $< \lambda$  nodes is included in  $Y_{<i}$  for some  $i < \lambda$ .

For this we shall prove more by induction on  $j < \lambda$ :

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<sup>1</sup>the guessing clubs are used only in Stage D.

$\oplus_{2,j}$  if  $i < j, i \notin S, \mathbf{c}_1$  a colouring of  $G|Y_{<i}, \text{Rang}(\mathbf{c}_1) \subseteq \kappa$  and  $u \in [\kappa]^\kappa$  then there is a colouring  $\mathbf{c}_2$  of  $G|Y_{<j}$  extending  $\mathbf{c}_1$  such that  $\text{Rang}(\mathbf{c}_2 \upharpoonright (Y_{<j} \setminus Y_{<i})) \subseteq u$ .

Case 1:  $j = 0$

Trivial.

Case 2:  $j$  successor,  $j - 1 \notin S$

By the induction hypothesis without loss of generality  $j = i + 1$ , but then every node from  $Y_j \setminus Y_i$  is an isolated node in  $G|Y_{<j}$ , because if  $\{(\alpha, \beta), (\alpha', \beta')\}$  is an edge of  $G|Y_j$  then  $\mathbf{i}(\beta), \mathbf{i}(\beta') \in S$  hence necessarily  $\mathbf{i}(\beta) \neq j - 1 = i, \mathbf{i}(\beta') \neq j - 1 = i$  hence both  $(\alpha, \beta), (\alpha, \beta')$  are from  $Y_i$ .

Case 3:  $j$  successor,  $j - 1 \in S$

Let  $j - 1$  be called  $\delta$  so  $\delta \in S$ . But  $i \notin S$  by the assumption in  $\oplus_{2,j}$  hence  $i < \delta$ . Let  $\varepsilon(*) < \kappa$  be such that  $\alpha_{\delta, \varepsilon(*)}^* > i$ .

Let  $\langle u_\varepsilon : \varepsilon \leq \kappa \rangle$  be a sequence of subsets of  $u$ , a partition of  $u$  to sets each of cardinality  $\kappa$ ; actually the only disjointness used is that  $u_\kappa \cap (\bigcup_{\varepsilon < \kappa} u_\varepsilon) = \emptyset$ .

We let  $i_0 = i, i_{1+\varepsilon} = \cup\{\alpha_{\delta, \varepsilon(*)+1+\zeta}^* + 1 : \zeta < 1 + \varepsilon\}, i_\kappa = \delta, i_{\kappa+1} = \delta + 1 = j$ .

Note that:

- $\varepsilon < \kappa \Rightarrow i_\varepsilon \notin S_j$ .

[Why? For  $\varepsilon = 0$  by the assumption on  $i$ , for  $\varepsilon$  successor  $i_\varepsilon$  is a successor ordinal and for  $i$  limit clearly  $\text{cf}(i_\varepsilon) = \text{cf}(\varepsilon) < \kappa$  and  $S \subseteq S_\kappa^\lambda$ .]

We now choose  $\mathbf{c}_{2,\zeta}$  by induction on  $\zeta \leq \kappa + 1$  such that:

- $\mathbf{c}_{2,0} = \mathbf{c}_1$
- $\mathbf{c}_{2,\zeta}$  is a colouring of  $G|Y_{<i_\zeta}$
- $\mathbf{c}_{2,\zeta}$  is increasing with  $\zeta$
- $\text{Rang}(\mathbf{c}_{2,\zeta} \upharpoonright (Y_{<i_{\zeta+1}} \setminus Y_{<i_\zeta})) \subseteq u_\xi$  for every  $\xi < \zeta$ .

For  $\zeta = 0, \mathbf{c}_{2,0}$  is  $\mathbf{c}_1$  so is given.

For  $\zeta = \varepsilon + 1 < \kappa$ : use the induction hypothesis, possible as necessarily  $i_\varepsilon \notin S$ .

For  $\zeta \leq \kappa$  limit: take union.

For  $\zeta = \kappa + 1$ , note that each node  $b$  of  $Y_{<i_\zeta} \setminus Y_{<i_\kappa}$  is not connected to any other such node and if the node  $b$  is connected to a node from  $Y_{<i_\kappa}$  then the node  $b$  necessarily has the form  $(\min(C_\delta), \gamma), \gamma \in X'_\delta$ , hence  $\bar{\beta}_\gamma$  is well defined, so the node  $b = (\min(C_\delta), \gamma)$  is connected in  $G$ , more exactly in  $G|Y_{\leq \delta}$  exactly to the  $\kappa$  nodes  $\{(\beta_{\gamma, 2\varepsilon}, \beta_{\gamma, 2\varepsilon+1}) : \varepsilon < \kappa\}$ , but for every  $\varepsilon < \kappa$  large enough,  $\mathbf{c}_{2,\kappa}((\beta_{\gamma, 2\varepsilon}, \beta_{\gamma, 2\varepsilon+1})) \in u_\varepsilon$  hence  $\notin u_\kappa$  and  $|u_\kappa| = \kappa$  so we can choose a colour.

Case 4:  $j$  limit

By the assumption of the claim there is a club  $e$  of  $j$  disjoint to  $S$  and without loss of generality  $\min(e) = i$ . Now choose  $\mathbf{c}_{2,\xi}$  a colouring of  $Y_{<\xi}$  by induction on  $\xi \in e \cup \{j\}$ , increasing with  $\xi$  such that  $\text{Rang}(\mathbf{c}_{2,\xi} \upharpoonright (Y_{<\xi} \setminus Y_{<i})) \subseteq u$  and  $\mathbf{c}_{2,0} = \mathbf{c}_1$

- For  $\xi = \min(e) = i$  the colouring  $\mathbf{c}_{2,\xi} = \mathbf{c}_{2,i} = \mathbf{c}_1$  is given,
- for  $\xi$  successor in  $e$ , i.e.  $\in \text{nacc}(e) \setminus \{i\}$ , use the induction hypothesis with  $\xi, \max(e \cap \xi)$  here playing the role of  $j, i$  there recalling  $\max(e \cap \xi) \in e, e \cap S = \emptyset$
- for  $\xi = \sup(e \cap \xi)$  take union.

Lastly, for  $\xi = j$  we are done.

Stage D:  $\text{ch}(G) > \kappa$ .

Why? Toward a contradiction, assume  $\mathbf{c}$  is a colouring of  $G$  with set of colours  $\subseteq \kappa$ . For each  $\gamma < \lambda$  let  $u_\gamma = \{\mathbf{c}((\alpha, \beta)) : \gamma < \alpha < \beta < \lambda \text{ and } (\alpha, \beta) \in Y\}$ . So  $\langle u_\gamma : \gamma < \lambda \rangle$  is  $\subseteq$ -decreasing sequence of subsets of  $\kappa$  and  $\kappa < \lambda = \text{cf}(\lambda)$ , hence for some  $\gamma(*) < \lambda$  and  $u_* \subseteq \kappa$  we have  $\gamma \in (\gamma(*), \lambda) \Rightarrow u_\gamma = u_*$ .

Hence  $E = \{\delta < \lambda : \delta \text{ is a limit ordinal } > \gamma(*) \text{ and } (\forall \alpha < \delta)(\mathbf{i}(\alpha) < \delta) \text{ and for every } \gamma < \delta \text{ and } i \in u_* \text{ there are } \alpha < \beta \text{ from } (\gamma, \delta) \text{ such that } (\alpha, \beta) \in Y \text{ and } \mathbf{c}((\alpha, \beta)) = i\}$  is a club of  $\lambda$ .

Now recall that  $\bar{C}$  guesses clubs hence for some  $\delta \in S$  we have  $C_\delta \subseteq E$ , so for every  $\varepsilon < \kappa$  we can choose  $\beta_{2\varepsilon} < \beta_{2\varepsilon+1}$  from  $(\alpha_{\delta, \varepsilon}^*, \alpha_{\delta, \varepsilon+1}^*)$  such that  $(\beta_{2\varepsilon}, \beta_{2\varepsilon+1}) \in Y$  and  $\varepsilon \in u_* \Rightarrow \mathbf{c}((\beta_{2\varepsilon}, \beta_{2\varepsilon+1})) = \varepsilon$ . So  $\langle \beta_\varepsilon : \varepsilon < \kappa \rangle$  is well defined, increasing and belongs to  $\Gamma_\delta$ , hence  $\bar{\beta}_\gamma = \langle \beta_\varepsilon : \varepsilon < \kappa \rangle$  for some  $\gamma \in X_\delta$ , hence  $(\alpha_{\delta, 0}^*, \gamma)$  belongs to  $Y$  and is connected in the graph to  $(\beta_{2\varepsilon}, \beta_{2\varepsilon+1})$  for  $\varepsilon < \kappa$ . Now if  $\varepsilon \in u_*$  then  $\mathbf{c}((\beta_{2\varepsilon}, \beta_{2\varepsilon+1})) = \varepsilon$  hence  $\mathbf{c}((\alpha_{\delta, 0}^*, \gamma)) \neq \varepsilon$  for every  $\varepsilon \in u_*$ , so  $\mathbf{c}((\alpha_{\delta, 0}^*, \gamma)) \in \kappa \setminus u_*$ . But  $u_* = u_{\alpha_{\delta, 0}^*}$  and  $\mathbf{c}((\alpha_{\delta, 0}^*, \gamma)) \in \kappa \setminus u_*$ , so we get contradiction to the definition of  $u_{\alpha_{\delta, 0}^*}$ .  $\square_{1.1}$

Similarly

**Claim 1.2.** *There is an increasing continuous sequence  $\langle G_i : i \leq \lambda \rangle$  of graphs each of cardinality  $\lambda^\kappa$  such that  $\text{ch}(G_\lambda) > \kappa$  and  $i < \lambda$  implies  $\text{ch}(G_i) \leq \kappa$  and even  $\text{cl}(G_i) \leq \kappa$  when:*

- $\boxplus$  (a)  $\lambda = \text{cf}(\lambda)$
- (b)  $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$  is stationary not reflecting.

*Proof.* Like 1.1 but the  $X_i$  are not necessarily  $\subseteq \lambda$  or use 2.2.  $\square_{1.2}$

## § 2. FROM ALMOST FREE

**Definition 2.1.** Suppose  $\eta_\beta \in {}^\kappa \text{Ord}$  for every  $\beta < \alpha(*)$  and  $u \subseteq \alpha(*)$ , and  $\alpha < \beta < \alpha(*) \Rightarrow \eta_\alpha \neq \eta_\beta$ .

1) We say  $\{\eta_\alpha : \alpha \in u\}$  is free when there exists a function  $h : u \rightarrow \kappa$  such that  $\langle \{\eta_\alpha(\varepsilon) : \varepsilon \in [h(\alpha), \kappa)\} : \alpha \in u \rangle$  is a sequence of pairwise disjoint sets.

2) We say  $\{\eta_\alpha : \alpha \in u\}$  is weakly free when there exists a sequence  $\langle u_{\varepsilon, \zeta} : \varepsilon, \zeta < \kappa \rangle$  of subsets of  $u$  with union  $u$ , such that the function  $\eta_\zeta \mapsto \eta_\zeta(\varepsilon)$  is a one-to-one function on  $u_{\varepsilon, \zeta}$ , for each  $\varepsilon, \zeta < \kappa$ .

**Claim 2.2.** 1) We have  $\text{INC}_{\text{chr}}(\mu, \lambda, \kappa)$  and even  $\text{INC}_{\text{chr}}^+(\mu, \lambda, \kappa)$ , see Definition 0.3(1), (5) when:

- ⊞ (a)  $\alpha(*) \in [\mu, \mu^+)$  and  $\lambda$  is regular  $\leq \mu$  and  $\mu = \mu^\kappa$
- (b)  $\bar{\eta} = \langle \eta_\alpha : \alpha < \alpha(*) \rangle$
- (c)  $\eta_\alpha \in {}^\kappa \mu$
- (d)  $\langle u_i : i \leq \lambda \rangle$  is a  $\subseteq$ -increasing continuous sequence of subsets of  $\alpha(*)$  with  $u_\lambda = \alpha(*)$
- (e)  $\bar{\eta} \upharpoonright u_\alpha$  is free iff  $\alpha < \lambda$  iff  $\bar{\eta} \upharpoonright u_\alpha$  is weakly free.

2) We have  $\text{INC}_{\text{chr}}[\mu, \lambda, \kappa]$  and even  $\text{INC}_{\text{chr}}^+[\mu, \lambda, \kappa]$ , see Definition 0.3(4) when:

- ⊞<sub>2</sub> (a), (b), (c) as in ⊞ from 2.2
- (d)  $\bar{\eta}$  is not free
- (e)  $\bar{\eta} \upharpoonright u$  is free when  $u \in [\alpha(*)]^{<\lambda}$ .

*Proof.* We concentrate on proving part (1); the proof of part (2) is similar. For  $\mathcal{A} \subseteq {}^\kappa \text{Ord}$ , we define  $\tau_{\mathcal{A}}$  as the vocabulary  $\{P_\eta : \eta \in \mathcal{A}\} \cup \{F_\varepsilon : \varepsilon < \kappa\}$  where  $P_\eta$  is a unary predicate,  $F_\varepsilon$  a unary function (will be interpreted as possibly partial).

Without loss of generality for each  $i < \lambda$ ,  $u_i$  is an initial segment of  $\alpha(*)$  and let  $\mathcal{A} = \{\eta_\alpha : \alpha < \alpha(*)\}$  and let  $<_{\mathcal{A}}$  be the well ordering  $\{(\eta_\alpha, \eta_\beta) : \alpha < \beta < \alpha(*)\}$  of  $\mathcal{A}$ .

We further let  $K_{\mathcal{A}}$  be the class of structures  $M$  such that (pedantically,  $K_{\mathcal{A}}$  depend also on the sequence  $\langle \eta_\alpha : \alpha < \alpha(*) \rangle$ ):

- ⊞<sub>1</sub> (a)  $M = (|M|, F_\varepsilon^M, P_\eta^M)_{\varepsilon < \kappa, \eta \in \mathcal{A}}$
- (b)  $\langle P_\eta^M : \eta \in \mathcal{A} \rangle$  is a partition of  $|M|$ , so for  $a \in M$  let  $\eta_a = \eta_a^M$  be the unique  $\eta \in \mathcal{A}$  such that  $a \in P_\eta^M$
- (c) if  $a_\ell \in P_{\eta_\ell}^M$  for  $\ell = 1, 2$  and  $F_\varepsilon^M(a_2) = a_1$  then  $\eta_1(\varepsilon) = \eta_2(\varepsilon)$  and  $\eta_1 <_{\mathcal{A}} \eta_2$ .

Let  $K_{\mathcal{A}}^*$  be the class of  $M$  such that

- ⊞<sub>2</sub> (a)  $M \in K_{\mathcal{A}}$
- (b)  $\|M\| = \mu$
- (c) if  $\eta \in \mathcal{A}$ ,  $u \subseteq \kappa$  and  $\eta_\varepsilon <_{\mathcal{A}} \eta$ ,  $\eta_\varepsilon(\varepsilon) = \eta(\varepsilon)$  and  $a_\varepsilon \in P_{\eta_\varepsilon}^M$  for  $\varepsilon \in u$  then for some  $a \in P_\eta^M$  we have  $\varepsilon \in u \Rightarrow F_\varepsilon^M(a) = a_\varepsilon$  and  $\varepsilon \in \kappa \setminus u \Rightarrow F_\varepsilon^M(a)$  not defined.

Clearly



$\boxplus_3$  there is  $M \in K_{\mathcal{A}}^*$ .

[Why? As  $\mu = \mu^\kappa$  and  $|\mathcal{A}| = \mu$ .]

$\boxplus_4$  for  $M \in K_{\mathcal{A}}$  let  $G_M$  be the graph with:

- set of nodes  $|M|$
- set of edges  $\{\{a, F_\varepsilon^M(a)\} : a \in |M|, \varepsilon < \kappa \text{ when } F_\varepsilon^M(a) \text{ is defined}\}$ .

Now

$\boxplus_5$  if  $u \subseteq \alpha(*)$ ,  $\mathcal{A}_u = \{\eta_\alpha : \alpha \in u\} \subseteq \mathcal{A}$  and  $\bar{\eta} \upharpoonright u$  is free, and  $M \in K_{\mathcal{A}}$  then  $G_{M, \mathcal{A}_u} := G_M \upharpoonright (\cup \{P_\eta^M : \eta \in \mathcal{A}_u\})$  has chromatic number  $\leq \kappa$ ; moreover has colouring number  $\leq \kappa$ .

[Why? Let  $h : u \rightarrow \kappa$  witness that  $\bar{\eta} \upharpoonright u$  is free and for  $\varepsilon < \kappa$  let  $\mathcal{B}_\varepsilon := \{\eta_\alpha : \alpha \in u \text{ and } h(\alpha) = \varepsilon\}$ , so  $\mathcal{B} = \cup \{\mathcal{B}_\varepsilon : \varepsilon < \kappa\}$ , hence it is enough to prove for each  $\varepsilon < \kappa$  that  $G_{\mu, \mathcal{B}_\varepsilon}$  has chromatic number  $\leq \kappa$ . To prove this, by induction on  $\alpha \leq \alpha(*)$  we choose  $\mathbf{c}_\alpha^\varepsilon$  such that:

- $\boxplus_{5.1}$  (a)  $\mathbf{c}_\alpha^\varepsilon$  is a function  
 (b)  $\langle \mathbf{c}_\beta : \beta \leq \alpha \rangle$  is increasing continuous  
 (c)  $\text{Dom}(\mathbf{c}_\alpha^\varepsilon) = B_\alpha^\varepsilon := \cup \{P_{\eta_\beta}^M : \beta < \alpha \text{ and } \eta_\beta \in \mathcal{B}_\varepsilon\}$   
 (d)  $\text{Rang}(\mathbf{c}_\alpha^\varepsilon) \subseteq \kappa$   
 (e) if  $a, b \in \text{Dom}(\mathbf{c}_\alpha)$  and  $\{a, b\} \in \text{edge}(G_M)$  then  $\mathbf{c}_\alpha(a) \neq \mathbf{c}_\alpha(b)$ .

Clearly this suffices. Why is this possible?

If  $\alpha = 0$  let  $\mathbf{c}_\alpha^\varepsilon$  be empty, if  $\alpha$  is a limit ordinal let  $\mathbf{c}_\alpha^\varepsilon = \cup \{\mathbf{c}_\beta^\varepsilon : \beta < \alpha\}$  and if  $\alpha = \beta + 1 \wedge \alpha(\beta) \neq G$  let  $\mathbf{c}_\alpha = \mathbf{c}_\beta$ .

Lastly, if  $\alpha = \beta + 1 \wedge h(\beta) = \varepsilon$  we define  $\mathbf{c}_\alpha^\varepsilon$  as follows for  $a \in \text{Dom}(\mathbf{c}_\alpha^\varepsilon)$ ,  $\mathbf{c}_\alpha^\varepsilon(a)$  is:

Case 1:  $a \in B_\beta^\varepsilon$ .

Then  $\mathbf{c}_\alpha^\varepsilon(a) = \mathbf{c}_\beta^\varepsilon(a)$ .

Case 2:  $a \in B_\alpha^\varepsilon \setminus B_\beta^\varepsilon$ .

Then  $\mathbf{c}_\alpha^\varepsilon(a) = \min(\kappa \setminus \{\mathbf{c}_\beta^\varepsilon(F_\zeta^M(a)) : \zeta < \varepsilon \text{ and } F_\zeta^M(a) \in \text{Dom}(\mathbf{c}_\beta^\varepsilon)\})$ .

This is well defined as:

- $\boxplus_{5.2}$  (a)  $B_\alpha^\varepsilon = B_\beta^\varepsilon \cup P_{\eta_\beta}^M$   
 (b) if  $a \in B_\beta^\varepsilon$  then  $\mathbf{c}_\beta^\varepsilon(a)$  is well defined (so case 1 is O.K.)  
 (c) if  $\{a, b\} \in \text{edge}(G_M)$ ,  $a \in P_{\eta_\beta}^M$  and  $b \in B_\alpha^\varepsilon$  then  $b \in B_\beta^\varepsilon$  and  $b \in \{F_\zeta^M(a) : \zeta < \varepsilon\}$   
 (d)  $\mathbf{c}_\alpha^\varepsilon(a)$  is well defined in Case 2, too  
 (e)  $\mathbf{c}_\alpha^\varepsilon$  is a function from  $B_\alpha^\varepsilon$  to  $\kappa$   
 (f)  $\mathbf{c}_\alpha^\varepsilon$  is a colouring.

[Why? Clause (a) by  $\boxplus_{5.1}(c)$ , clause (b) by the induction hypothesis and clause (c) by  $\boxplus_1(c) + \boxplus_4$ . Next, clause (d) holds as  $\{\mathbf{c}_\beta^\varepsilon(F_\zeta^M(a)) : \zeta < \varepsilon \text{ and } F_\zeta^M(a) \in B_\beta^\varepsilon = \text{Dom}(\mathbf{c}_\beta^\varepsilon)\}$  is a set of cardinality  $\leq |\varepsilon| < \kappa$ . Clause (e) holds by the choices of the  $\mathbf{c}_\alpha^\varepsilon(a)$ 's. Lastly, to check that clause (f) holds assume  $(a, b)$  is an edge of  $G_M \restriction B_\alpha^\varepsilon$ , for some  $\zeta < \kappa$  we have  $b = F_\zeta^M(a)$ , hence  $\eta_a^M <_{\mathcal{A}} \eta_b^M$ . If  $a, b \in B_\beta^\varepsilon$  use the induction hypothesis. Otherwise,  $\zeta < \varepsilon$  by the definition of “ $h$  witnesses  $\bar{\eta} \restriction u$  is free” and the choice of  $B_\alpha^\varepsilon$  in  $\boxplus_{5.1}(c)$ . Now use the choice of  $\mathbf{c}_\alpha^\varepsilon(a)$  in Case 2 above.]

So indeed  $\boxplus_5$  holds.]

$\boxplus_6$   $\text{chr}(G_M) > \kappa$  if  $M \in K_{\mathcal{A}}^*$ .

Why? Toward contradiction assume  $\mathbf{c} : G_M \rightarrow \kappa$  is a colouring. For each  $\eta \in \mathcal{A}$  and  $\varepsilon < \kappa$  let  $\Lambda_{\eta, \varepsilon} = \{\nu : \nu \in \mathcal{A}, \nu <_{\mathcal{A}} \eta, \nu(\varepsilon) = \eta(\varepsilon) \text{ and for some } a \in P_\nu^M \text{ we have } \mathbf{c}(a) = \varepsilon\}$ .

Let  $\mathcal{B}_\varepsilon = \{\eta \in \mathcal{A} : |\Lambda_{\eta, \varepsilon}| < \kappa\}$ . Now if  $\mathcal{A} \neq \bigcup \{\mathcal{B}_\varepsilon : \varepsilon < \kappa\}$  then pick any  $\eta \in \mathcal{A} \setminus \bigcup \{\mathcal{B}_\varepsilon : \varepsilon < \kappa\}$  and by induction on  $\varepsilon < \kappa$  choose  $\nu_\varepsilon \in \Lambda_{\eta, \varepsilon} \setminus \{\nu_\zeta : \zeta < \varepsilon\}$ , possible as  $\eta \notin \mathcal{B}_\varepsilon$  by the definition of  $\mathcal{B}_\varepsilon$ . By the definition of  $\Lambda_{\eta, \varepsilon}$  there is  $a_\varepsilon \in P_{\nu_\varepsilon}^M$  such that  $\mathbf{c}(\nu_\varepsilon) = \varepsilon$ . So as  $M \in K_{\mathcal{A}}^*$  there is  $a \in P_\eta^M$  such that  $\varepsilon < \kappa \Rightarrow F_\varepsilon^M(a) = a_\varepsilon$ , but  $\{a, a_\varepsilon\} \in \text{edge}(G_M)$  hence  $\mathbf{c}(a) \neq \mathbf{c}(a_\varepsilon) = \varepsilon$  for every  $\varepsilon < \kappa$ , contradiction. So  $\mathcal{A} = \bigcup \{\mathcal{B}_\varepsilon : \varepsilon < \kappa\}$ .

For each  $\varepsilon < \kappa$  we choose  $\zeta_\eta < \kappa$  for  $\eta \in \mathcal{B}_\varepsilon$  by induction on  $<_{\mathcal{A}}$  such that  $\zeta_\eta \notin \{\zeta_\nu : \nu \in \Lambda_{\eta, \varepsilon} \cap \mathcal{B}_\varepsilon\}$ . Let  $\mathcal{B}_{\varepsilon, \zeta} = \{\eta \in \mathcal{B}_\varepsilon : \zeta_\eta = \zeta\}$  for  $\varepsilon, \zeta < \kappa$  so  $\mathcal{A} = \bigcup \{\mathcal{B}_{\varepsilon, \zeta} : \varepsilon, \zeta < \kappa\}$  and clearly  $\eta \mapsto \eta(\varepsilon)$  is a one-to-one function with domain  $\mathcal{B}_{\varepsilon, \zeta}$ , contradiction to “ $\bar{\eta} = \eta \restriction u_\lambda$  is not weakly free”.  $\square_{2.2}$

**Observation 2.3.** 1) If  $\mathcal{A} \subseteq {}^\kappa \mu$  and  $\eta \neq \nu \in \mathcal{A} \Rightarrow (\forall^\infty \varepsilon < \kappa)(\eta(\varepsilon) \neq \nu(\varepsilon))$  then  $\mathcal{A}$  is free iff  $\mathcal{A}$  is weakly free.

2) The assumptions of 2.2(2) hold when:  $\mu \geq \lambda > \kappa$  are regular,  $S \subseteq S_\kappa^\mu$  stationary,  $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ ,  $\eta_\delta$  an increasing sequence of ordinals of length  $\kappa$  with limit  $\delta$  such that  $u \subseteq [\lambda]^{<\lambda} \Rightarrow \langle \text{Rang}(\eta_\delta) : \eta \in u \rangle$  has a one-to-one choice function.

**Conclusion 2.4.** Assume that for every graph  $G$ , if  $H \subseteq G \wedge |H| < \lambda \Rightarrow \text{chr}(H) \leq \kappa$  then  $\text{chr}(G) \leq \kappa$ .

Then:

- (A) if  $\mu > \kappa = \text{cf}(\mu)$  and  $\mu \geq \lambda$  then  $\text{pp}(\mu) = \mu^+$
- (B) if  $\mu > \text{cf}(\mu) \geq \kappa$  and  $\mu \geq \lambda$  then  $\text{pp}(\mu) = \mu^+$ , i.e. the strong hypothesis
- (C) if  $\kappa = \aleph_0$  then above  $\lambda$  the SCH holds.

*Proof.* Clause (A): By 2.2 and [Sh:g, Ch.II], [Sh:g, Ch.IX, §1].

Clause (B): Follows from (A) by [Sh:g, Ch.VIII, §1].

Clause (C): Follows from (B) by [Sh:g, Ch.IX, §1].  $\square_{2.4}$

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